Lecture 25

In this lecture, we'll use the orbit-stabilizer theorem to obtain a very nice equation, called the Class equation for counting the # of elements in a group, which has far reaching applications.

Being in the same orbit is an equivalence relation

Let G be a group acting on a set X. Recall that for $x \in X$, $O_x = \{g \cdot x \mid g \in G\}$. If $y \in X$ is also in O_x then $\exists\ g \in G$ s.t. $y = g \cdot x$.

Define a relation $\sim$ on X by
$x \sim y$ iff $y = g \cdot x$ for some $g \in G$, i.e., $y \in O_x$.

Proposition 1:- The relation $\sim$ on X defined above is an equivalence relation.

<u>Proof</u> :- Recall that for proving a relation $\sim$ to be an equivalence relation, we need to check 3 things which we do below.

1) Reflexive :- $x \sim x$. Choose $e \in G$. Then

$$e \cdot x = x \Rightarrow x \sim x.$$

2) Symmetric :- If $x \sim y \Rightarrow y \sim x$.

If $x \sim y \Rightarrow y = g \cdot x$ for some $g \in G$.

Act both sides by g^{-1} to get $g^{-1} \cdot y = g^{-1} \cdot (g \cdot x)$
$= e \cdot x = x$

So $x = g^{-1} \cdot y \Rightarrow y \sim x$

3) Transitive :- If $x \sim y$, $y \sim z \Rightarrow x \sim z$.

If $x \sim y \Rightarrow \exists\, g \in G$ s.t. $y = g \cdot x$

$y \sim z \Rightarrow \exists\, h \in G$ s.t. $z = h \cdot y$

$\therefore \quad z = h\cdot(g\cdot x) = (hg)\cdot x$ [by the definition of a group action.]

$\Rightarrow \quad x \sim z$

So $\sim$ is an equivalence relation. ☑

The good thing about equivalence relations is that the equivalence classes partitions the set. Recall that a equivalence class of an equivalence relation $\sim$ is

$[x] = \{ y \in X \mid x \sim y \}$

So for the relation we defined on X, the equivalence class of x is just O_x.

So, $\{O_x\}_{x\in X}$ partitions the set X, i.e., if O_x and O_y are two orbits then either

$$O_x = O_y \quad \text{or}$$
$$O_x \cap O_y = \phi$$

Remark :- Notice that O_x have the same properties as cosets of a subgroup. This is because the relation on G of being in the same coset is an equivalence relation and the equivalence classes are just cosets.

Since $\{O_x\}_{x \in X}$ partitions $X \Rightarrow$ If $y \in X$ is any element then there is one and only one element $z \in X$ s.t. $y \in O_z$.

So if X is finite then

$$|X| = \sum_{\substack{i=1 \\ x_i \in X}}^{n} |O_{x_i}| \qquad \text{——— } ①$$

Note that ① holds for any group G acting on any set X. We'll use this to our advantage by looking at a particular action.

Conjugacy Classes and Class Equation

Recall that a group G act on itself by conjugation i.e, for $g, h \in G$, $g \cdot h = ghg^{-1}$.

We saw that for this action,

$$\mathrm{Stab}(g) = C(g) \quad \forall\, g \in G. \quad \text{—— } (2)$$

Definition :- Let G act on itself by conjugation. Then for $g \in G$, O_g is called the conjugacy class of g.

So a conjugacy class is just another name for an orbit.

Suppose $g \in Z(G)$. Let's see what the conjug-
-acy class of g is. By definition,

$$\begin{aligned} O_g &= \{ h \cdot g \mid h \in G \} \\ &= \{ hgh^{-1} \mid h \in G \} \\ &= \{ ghh^{-1} \mid h \in G \} \quad [\text{as } g \in Z(G)] \\ &= \{ g \} \end{aligned}$$

So for the conjugation action, $\forall g \in Z(G)$, $O_g = \{g\}$. Moreover, we can retrace our steps in the above process, i.e., if $O_g = \{g\} \Rightarrow g \in Z(G)$. Thus we get

<u>Proposition 2</u> Let $G \circlearrowleft G$ by conjugation. Then $O_g = \{g\} \Leftrightarrow g \in Z(G)$.

Let's get back to eq. 1 for the conjugation action. If G is finite then (since $X = G$ here)

$$|G| = \sum_{\substack{i=1 \\ g_i \in G}}^{n} |O_{g_i}|$$

But all those $g_i \in G$ which are in $Z(G)$, contribute only 1 element, g_i itself, so we can take $|Z(G)|$ many elements out to get

$$|G| = |Z(G)| + \sum_{\substack{j=1 \\ g_j \notin Z(G)}}^{k} |O_{g_j}| \quad \text{— } ③$$

Recall from the Orbit-Stabilizer theorem that if $|G| < \infty \Rightarrow |G| = |O_g||Stab(g)|$

So from ②, $|G| = |O_g||C(g)|$. $\Rightarrow |O_g| = \dfrac{|G|}{|C(g)|}$

Thus, using this in equation ③ we get

$$\boxed{|G| = |Z(G)| + \sum_{g \notin Z(G)} \frac{|G|}{|C(g)|}}$$

This is known as the class equation. An extraor-dinary thing about this is that even though we started w/ the conjugation action but the class equation holds for any finite group G, irresp-ective of any action.

This is the power of group actions! Even though we start w/ a particular action, suited to our needs, the end result holds for any group and we get a very nice general result.